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Spinor Fields in Non-Abelian Klein–Kaluza Theories

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We introduce a dimensional reduction procedure for a spinor field and a generalization of the minimal coupling scheme. We get an electric dipole moment of fermions of value 10^{-31} cm and PC breaking for a gauge group G with odd parameters. Reflection in higher (additional) dimensions are proposed as a conjugation of "color" charges connected with Yang–Mills fields. Our approach cancels Planck's mass terms in the Dirac equation.

1. INTRODUCTION

In this paper we deal with spinor fields in the framework of non-Abelian Klein–Kaluza theories. We generalize methods and results from Thirring (1972) and Kalinowski (1981a, 1981b) to a non-Abelian case. To do this we introduce on P an $(n+4)$-dimensional Klein–Kaluza manifold (Kerner, 1968; Cho, 1975; Kalinowski, 1983) a spinor field belonging to the fundamental representation of $SO(1,n+3)$. We assume that this spinor field depends on group coordinates in a trivial way, i.e., by the action of group G (G is a gauge group of Yang–Mills fields which we combine with gravity in the Klein–Kaluza framework).

We introduce for this spinor field new kinds of gauge derivatives. These gauge derivatives were defined in Kalinowski (1981a, 1981b) in a five-dimensional (electromagnetic) case. We generalize here this approach. Simultaneously we define a dimensional reduction procedure for spinor fields. It contains three steps:

1. We take a section of the bundle P and apply it for a spinor field Ψ;
2. We restrict $SO(1,n+3)$ to $SO(1,3)$ for Ψ;

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3. We decompose Ψ to spinor fields belonging to the Dirac representation of $SL(2,\mathbb{C})$.

After this we get on space-time E a tower of $2^{[n/2]}$ fermions fields.

In Kalinowski (1981a, 1981b) one introduces a similar construction for the five-dimensional (electromagnetic) case. Here we clarify this construction as a kind of dimensional reduction.

After this we generalize a minimal coupling scheme for a spinor field Ψ. We define on P an $(n+4)$-dimensional manifold, a Lagrangian form. In this Lagrangian we substitute the new gauge derivative for the field Ψ. This procedure is a simple generalization of that form (Kalinowski, 1981b). In the Lagrangian we obtain a new term similar to that from Kalinowski (1981b). In Kalinowski (1981b) such a term was interpreted as an interaction of an electric fermion dipole moment with the electromagnetic field. Here the interpretation is more complex. If we perform the dimensional reduction procedure we get on E (space-time) a sum of Lagrangians for all fermions from a tower describing interaction of these fermions with gravity and Yang–Mills fields in the usual way plus new terms. These new terms describe interactions of the Yang–Mills fields with fermions from a tower. If the number of group parameters is odd ($\dim G = 2l+1$) some of these terms may be interpreted as an interaction of fermion electric dipole moments with the electromagnetic field. In the case of even parameters of group G ($\dim G = n = 2l$) such terms are absent. Thus a fermion electric dipole moment is possible only in the case of odd parameters. But apart from these terms we have also other terms. These terms may be treated as anomalous dipole moments for "magnetic" parts of the Yang–Mills field. In the case of odd parameter groups we have PC breaking. This breaking is obviously very small because the value of fermion dipole electric moments is about 10^{-31} [cm] q. Similarly as in Kalinowski (1981b) this value is built only from fundamental constants.

In the paper we define also discrete transformations on P and interpret them as operators of parity, time-reversal, charge conjugations, PC, and PCT. Charge conjugations are defined as reflections in n additional dimensions (gauge dimensions). The paper is organized as follows. In the first section we describe some elements of the non-Abelian Klein–Kaluza theory and define geometric quantities which we use all along in the paper. In the second section we introduce a dimensional reduction procedure. In the third we introduce gauge derivatives of new kinds for a spinor field Ψ and generalize the minimal coupling scheme. We get here new terms in the Lagrangian. In the fourth section discrete transformations for a spinor field Ψ on P are defined. In the Appendix we deal with elements of the Clifford algebras which we use in the paper.

2. THE KLEIN–KALUZA THEORY

Let us introduce the principal fiber bundle P over the space-time E with the structural group G and with the projection π and let ω be a connection form on P.

Let us suppose that (E, g) is a manifold with a metric tensor g and Riemann connection $\bar{\omega}_{\alpha\beta}$, where $g = g_{\alpha\beta}\bar{\theta}^{\alpha}\otimes\bar{\theta}^{\beta}$. The signature of g is $(- - - +)$ and $\bar{\theta}^{\alpha}$ is a frame on E. Let us introduce a natural frame on P.

$$\theta^{A} = \left(\pi^{*}(\bar{\theta}^{\alpha}), \theta^{a} = \lambda\omega^{a}\right) \ \lambda > 0, \qquad \text{const} \tag{1}$$

$\omega = \omega^{a}X_{a}$ is a connection on P. The two-form of curvature of connection ω is

$$\Omega = \operatorname{hor} d\,\omega = (1/2)H^{a}_{\mu\nu}\theta^{\mu}\wedge\theta^{\nu}X_{a} \tag{2}$$

Ω obeys the structural Cartan's equation:

$$\Omega = d\omega + 1/2[\omega, \omega] \tag{3}$$

Bianchi's identity for ω is

$$\operatorname{hor} d\,\Omega = 0 \tag{4}$$

The map $e\colon E \supset \to P$, so that $e\cdot\pi = \mathrm{id}$ is called a cross section. From the physical point of view it means a particular choice of gauge. Thus

$$\begin{aligned} e^{*}\omega &= e^{*}(\omega^{a}X_{a}) = A^{a}_{\mu}\bar{\theta}^{\mu}X_{a} \\ e^{*}\Omega &= e^{*}(\Omega^{a}X_{a}) = 1/2F^{a}_{\mu\nu}\bar{\theta}^{\mu}\wedge\bar{\theta}^{\nu}X_{a} \end{aligned} \tag{5}$$

where

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} - C^{a}_{bc}A^{b}_{\nu}A^{c}_{\mu} \tag{6}$$

X_{a}, $a = 1, 2, \ldots \dim G = n$ are generators of the Lie algebra of group G and $[X_{a}, X_{b}] = C^{c}_{ab}X_{c}$.

A covariant derivative on P with respect to ω, d_{1} is defined as follows:

$$d_{1}\Psi = \operatorname{hor} d\,\Psi \tag{7}$$

This derivative is called a "gauge" derivative, where Ψ is, for example, a spinor field on P.

It is convenient to introduce the following notations. Capital latin indices A, B, C run $1,2,3,4,\ldots,n+4$, $\dim G = n$. Lower case greek indices $\alpha, \beta, \gamma, \delta = 1,2,3,4$ and lower case latin cases $a, b, c, d = 5,6,\ldots,n+4$. The symbol "$^{-}$" over θ^α and $\omega_{\alpha\beta}$ (i.e., $\bar{\theta}^\alpha$, $\bar{\omega}_{\alpha\beta}$) indicates that both quantities are defined on E.

Let us introduce now a tensor $\gamma = \gamma_{AB}\theta^A \otimes \theta^B$ on the manifold P in the natural way (Trautman, 1970, 1971b, 1973a) let $X, Y \in T_{\tan}(P)$.

$$\gamma(X, Y) = g(\pi' X, \pi' Y) + h_{ab}\theta^a(x)\theta^b(Y) \quad \text{or}$$

$$\gamma = \pi^* g + h_{ab}\theta^a \otimes \theta^b \tag{8}$$

Tensor γ has signature $(--+\underbrace{--\cdots-}_{n\ \text{times}})$.

$h_{ab} = C^c_{ad}C^d_{cb}$ is a Killing's tensor on G. In this frame this tensor has the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & h_{ab} \end{array}\right) \tag{9}$$

It is clear that the frame θ^A is partially unholomical, because

$$d\theta^a = \lambda\left(\Omega^a - \frac{1}{2\lambda^2}C^a_{bc}\theta^b \wedge \theta^c\right) \neq 0 \tag{10}$$

We also introduce a dual frame

$$\gamma(\xi_A) = \gamma_{AB}\theta^B \tag{11}$$

We have $\xi_A = (\xi_\alpha, \xi_a)$ and according to Trautman (1970)

$$\underset{\xi_a}{\mathfrak{L}}\gamma = 0 \tag{12}$$

Thus ξ_a are Killing's vectors of metric γ. Let us define now the Riemannian connection ω_{AB} on P and the exterior covariant derivative D with respect to ω_{AB}

$$D\gamma_{AB} = 0 \qquad \text{and} \qquad D\theta^A = 0 \tag{13}$$

The solution of (13) is

$$\omega_{\alpha\beta} = \pi^*(\bar{\omega}_{\alpha\beta}) - \frac{\lambda}{2} H_{\alpha\beta a}\theta^a$$

$$\omega_{\alpha b} = -\omega_{b\alpha} = -\frac{\lambda}{2} H_{\alpha\gamma b}\theta^\gamma \tag{14}$$

$$\omega_{ab} = -\omega_{ba} = -\frac{1}{2\lambda} C_{abc}\theta^c$$

ω_{AB} is invariant with respect to the action of group G (Cho, 1975). In Kalinowski (1981a, 1981b) we introduce new kinds of gauge derivatives for a spinor field Ψ. Because of the derivatives we avoided some troubles which appeared in Thirring (1972). We get for the electromagnetic case $[G = U(1)]$ the fermion electric dipole moment without the Planck's mass term in the Dirac equation. In the Klein–Kaluza theory $\lambda = 2\varepsilon(\sqrt{G}/c^2)$, $\varepsilon^2 = 1$, where G is the gravitational constant and c is a velocity of light in vacuum. This condition originates from the consistency between the equation in the Klein–Kaluza theory and Einstein equation (Kaluza, 1921; Lichnerowicz, 1955a; Kerner, 1968). It is worth noting that this condition does not determine the sign of λ. It was unnoticed in Thirring (1972) and Kalinowski (1981a, 1981b). Now we define the dual Cartan's base on E. Let $\eta_{1234} = (-\det g)^{1/2}$ and $\eta_{\alpha\beta\gamma\delta}$ is the Levi-Cività symbol and

$$\eta_\alpha = \tfrac{1}{6}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma \wedge \bar{\theta}^\delta \eta_{\alpha\beta\gamma\delta}$$
$$\eta = 1/4\bar{\theta}^\alpha \wedge \eta_\alpha \tag{15}$$

Details concerning elements of geometry mentioned here can be found in Trautman (1970, 1971, 1980), Kobayashi et al. (1963), and Lichnerowicz (1955b).

3. DIMENSIONAL REDUCTION

Let us consider the group $SO(1, n+3)$ and its fundamental (complex) representation of dimension $K = 4\cdot 2^{[n/2]}$, where $[n/2] = 1$ for $n = 2l$ or $2l+1$:

$$U(g)\Psi(X) = D^F(g)\Psi(g^{-1}X) \qquad X \in M^{(1,n+3)}, \qquad g \in SO(1, n+3) \tag{16}$$

$SO(1, n+3)$ acts linearly in $M^{(1,n+3)}$ [$(n+4)$-dimensional Minkowski space]. The Lorentz group $SO(1,3) \subset SO(1, n+3)$. Thus after restriction g to subgroup $SO(1,3)$ we obtain a decomposition of D^F (Barut et al., 1977) according to

$$D^F|_{SO(1,3)}(\Lambda) = \underbrace{L(\Lambda) \oplus \cdots \oplus L(\Lambda)}_{[n/2] \text{ times}}, \qquad \Lambda \in SO(1,3) \tag{17}$$

where

$$L(\Lambda) = D^{(1/2,0)} \oplus D^{(0,1/2)}(\Lambda)$$

is the Dirac representation of $SO(1,3)$. The decomposition (17) for a spinor Ψ has a form

$$\Psi|_{SO(1,3)} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{2\left[\frac{n}{2}\right]} \end{bmatrix} \tag{18}$$

where ψ_i, $i = 1, 2, \ldots, 2^{[n/2]}$ are spinors belonging to the Dirac representation $(L = D^{(1/2,0)} \oplus D^{(0,1/2)})$. Thus, owing to the decomposition (18) we get a tower of $1/2$ spin fermions.

More precisely, we deal with representations of Spin$(1, n+3)$ and Spin $(1,3) \approx SL(2, \mathbb{C})$.

Let us turn to a manifold P. It is a metric manifold (P, γ) with a metric tensor γ. At every point p element P a tangent space $T_p(P) \simeq M^{(1,n+3)}$. Let $\Psi: P \to \mathbb{C}^K$ $(K = 2^{[n/2]})$ be a spinor field on P at every point $p \in P$ belonging to fundamental representation D^F of group $SO(1, n+3)$.

For spinor field Ψ we suppose the following action of group G:

$$\Psi(pg_1) = \sigma(g_1^{-1})\Psi(p) \tag{19}$$

where $p = (x, g) \in P$, g, $g_1 \in G$. σ is a representation of group G in $4.2^{[n/2]}$-dimensional complex space.

If we take a section $e: E \to P$ we get a spinor field $\Psi(e(x))$ on the manifold E (space-time). Thus it means that at every point $x \in E$ we have after restriction to $SO(1,3)$ spinor $\Psi|_{SO(1,3)}$ and for it the decomposition

(18) is valid. Thus

$$(e^*\Psi)_{|SO(1,3)}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi 2^{[n/2]}(x) \end{pmatrix} \tag{20}$$

Spinor fields $\Psi_i(x)$, $i = 1, 2, \ldots, 2^{[n/2]}$ are spinor fields at every point $x \in E$ belonging to the Dirac representation $L = D^{(0,1/2)} \oplus D^{(1/2,0)}$. Such a procedure we will call the dimensional reduction for a spinor field. In this way we get a tower of Dirac spinor fields on E. The following graph symbolizes it:

$$\Psi \underset{\text{section of } P}{\longrightarrow} e^*\Psi \underset{\substack{\text{from } SO(1,n+3) \\ \text{to } SO(1,3)}}{\overset{\text{restriction}}{\longrightarrow}} e^*\Psi|_{SO(1,3)} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi 2^{[n/2]} \end{pmatrix} \tag{21}$$

In Kalinowski (1981a, b) we dealt with (in a similar context) five-dimensional (electromagnetic) case [$G = U(1)$, $n = 1$]. Thus we have the de Sitter group $SO(1,4)$ and we dealt with spinor Ψ belonging to fundamental representation of group Spin$(1,4) \simeq$ Sp(4). But for this case we have dim D^F = dim $D^F_{|SO(1,3)}$ and after dimensional reduction we get only one spinor field on E. The procedure (21) explains a construction given in Kalinowski (1981a,b). This procedure points out how to obtain a set of Dirac spinor fields ψ_i on E if one has a spinor field on P (with a special dependence of higher-group dimensions). But from the physical point of view more interesting is the opposite case. Really we have several spinor fields on E with which we connect physical fermion fields. From time to time it is possible to build a tower from these physical spinor fields. There were some attempts in constructing such towers (Kerner, 1980; Palla, 1978; Pati, 1980). Thus from physical point of view it would be interesting to describe physical fermions as a spinor field on P belonging to a fundamental representation of $SO(1, n+3)$ [Spin$(1, n+3)$]. Maybe it helps us in understanding of the generations of fermions. Now it is difficult to proceed because a group G (gauge group for Grand Unified Theories) is not well established and one suspects there are possibly many new generations. We know from an asymptotic freedom argument in Quantum Chromodynamics that a number of generations may be smaller than 9 (greater than 2).

4. SPINOR FIELD ON P

Let Ψ be a spinor field on P belonging to fundamental representation D^F of $SO(1, n+3)$ [Spin$(1, n+3)$] and let Γ^A, $A = 1, 2, \ldots, n+4$ be a representation of the Clifford algebra for $SO(1, n+3)$ acting in the space representation of D^F, i.e., $\Gamma^A \in C(1, n+3)$

$$\{\Gamma_A, \Gamma_B\} = 2\bar{g}_{AB}, \qquad \Gamma^A \in L(\mathbb{C}^k) \tag{22}$$

$K = 4.2^{[n/2]}$, $[n/2] = l$, where $\bar{g}_{AB} = \text{diag}(-1, -1, -1, 1\underbrace{-1, \ldots, -1}_{n \text{ times}})$. We introduce a spinor field $\bar{\Psi}$:

$$\bar{\Psi} = \Psi^+ B \tag{23}$$

where "+" is Hermitian conjugation and

$$\Gamma^{\alpha^+} = B\Gamma^\alpha B^{-1} \tag{24}$$

It is easy to see that

$$\bar{\Psi}(pg_1) = \bar{\Psi}(p)\sigma(g_1) \tag{25}$$

where $P \in (X, g) \in P$, $g, g_1 \in G$, σ is a unitary representation of group G acting in $4 \cdot 2^{[n/2]}$-dimensional complex space, $\sigma \in L(\mathbb{C}^k)$. Fields Ψ and $\bar{\Psi}$ are defined on P and P is assumed to have an orthogonal coordinate system θ^A. This coordinate system is in general nonholonomic. We perform an infinitesimal change of frame θ^A:

$$\theta^{A'} = \theta^A + \delta\theta^A = \theta^A - \varepsilon^A_B \theta^B, \qquad \varepsilon_{AB} + \varepsilon_{BA} = 0 \tag{26}$$

Suppose that field Ψ corresponds to θ^A and Ψ' to $\theta^{A'}$, then we get:

$$\begin{aligned} \Psi' &= \Psi + \delta\Psi = \Psi - \varepsilon^{AB}\hat{\sigma}_{AB}\Psi \\ \bar{\Psi}' &= \bar{\Psi} + \delta\bar{\Psi} = \bar{\Psi} + \bar{\Psi}\hat{\sigma}_{AB}\varepsilon^{AB} \end{aligned} \tag{27}$$

where $\hat{\sigma}_{AB} = \frac{1}{8}[\Gamma_A, \Gamma_B]$. Now we consider covariant derivatives of spinor fields Ψ and $\bar{\Psi}$ on P with respect to ω_{AB}. We get

$$\begin{aligned} D\Psi &= d\Psi + \omega^{AB}\hat{\sigma}_{AB}\Psi \\ D\bar{\Psi} &= d\bar{\Psi} - \omega^{AB}\bar{\Psi}\hat{\sigma}_{AB} \end{aligned} \tag{28}$$

In Kalinowski (1981a, b) one introduces new kinds of "gauge" derivatives for the five-dimensional case. Now we generalize the approach to an arbitrary gauge group G:

$$\begin{aligned} \mathbf{D}\Psi &= \operatorname{hor} D\Psi = d_1\Psi + \operatorname{hor}(\omega^{AB})\hat{\sigma}_{AB}\Psi \\ \mathbf{D}\overline{\Psi} &= \operatorname{hor} D\overline{\Psi} = d_1\overline{\Psi} - \operatorname{hor}(\omega^{AB})\overline{\Psi}\hat{\sigma}_{AB} \end{aligned} \tag{29}$$

Horizontality is understood in the sense of a connection ω on a bundle P. Using (4) one gets

$$\begin{aligned} \mathbf{D}\Psi &= \overline{\mathbf{D}}\Psi - \frac{\lambda}{8} H^{ab}_{\gamma}[\Gamma_a, \Gamma_b]\Psi\theta^{\gamma} \\ \mathbf{D}\overline{\Psi} &= \overline{\mathbf{D}\Psi} + \frac{\lambda}{8} H^{ab}_{\gamma}\overline{\Psi}[\Gamma_a, \Gamma_b]\theta^{\gamma} \end{aligned} \tag{30}$$

where

$$\begin{aligned} \overline{\mathbf{D}}\Psi &= \operatorname{hor} \mathbf{D}\Psi \\ \overline{\mathbf{D}\Psi} &= \operatorname{hor} D\overline{\Psi} \end{aligned} \tag{31}$$

D is an exterior covariant derivative with respect to $\bar{\omega}_{\alpha\beta}$ (on E). $\overline{D}$ is the normal gauge derivative and the generally covariant derivative with respect to $\bar{\omega}_{\alpha\beta}$. It describes the well-known minimal coupling scheme between spinor field Ψ, the gravitational field, and Yang–Mills fields. It is easy to see that these new "gauge" derivatives induce on P a new connection

$$\hat{\omega}_{AB} = \operatorname{hor}(\omega_{AB}) \tag{32}$$

We work with $\hat{\omega}_{AB}$ rather than ω_{AB}. In Kalinowski (1981a, b), because of these gauge derivatives one gets a fermion electric dipole moment and avoided well-known troubles (Thirring, 1972) (Planck's mass term in the Dirac equation). The connection $\hat{\omega}_{AB}$ has many interesting properties. In Kalinowski (1983) it was proved that the scalar of curvature for $\hat{\omega}_{AB}$ is the sum of the scalar curvature for $\bar{\omega}_{\alpha\beta}$ (on E) and $-(\lambda^2/4)h_{ab}F^{a\mu\nu}F^{b}_{\mu\nu}$ (Lagrangian of Yang–Mills field for gauge group G). For ω_{AB} we get additionally an enormous cosmological term (Cho, 1975). For Dirac fields on E we have the Lagrange 4-form (Trautman, 1973; Kalinowski, 1981a, 1981b):

$$L_D(\psi, \bar{\psi}, d) = \frac{i\hbar c}{2}(\bar{\psi} l \wedge d\psi + d\bar{\psi} \wedge l\psi) + m\bar{\psi}\psi\eta \tag{33}$$

where $l = \gamma_\mu \eta^\mu$.

Now we pass from ψ, $\bar{\psi}$ to Ψ, $\overline{\Psi}$ and from d to D. In this way one generalizes the minimal coupling scheme. Classically we should pass from d to d_1. Thus one easily writes

$$L_D(\Psi,\overline{\Psi},D)=\frac{i\hbar c}{2}(\overline{\Psi}l\wedge D\Psi+D\overline{\Psi}\wedge l\Psi)+m\overline{\Psi}\Psi\eta,\ \text{where } l=\Gamma_\mu\eta^\mu \quad (34)$$

Using (30) one easily gets

$$L_D(\Psi,\overline{\Psi},D)=L_D(\Psi,\overline{\Psi},\overline{D})+i\varepsilon\frac{\sqrt{G}\,\hbar}{4c}H_b^{\alpha\gamma}\overline{\Psi}\Gamma^b\left[\Gamma_\alpha,\Gamma_\gamma\right]\Psi\eta \quad (35)$$

and

$$2\varepsilon\frac{\sqrt{G}\,\hbar}{C}=2\frac{\varepsilon l_{\mathrm{Pl}}q}{\sqrt{\alpha}}\approx\pm0.95\times10^{-31}\,[\mathrm{cm}]q \quad (36)$$

where l_{Pl} is Planck's length, α the fine structure constant, q elementary charge, and $\varepsilon^2=1$.

If one performs the dimensional reduction (21) for $L_D(\Psi,\overline{\Psi},\overline{D})$ one easily gets (see Appendix A)

$$L_D(\Psi,\overline{\Psi},\overline{D})\underset{\substack{\text{dimensional}\\ \text{reduction}}}{\rightarrow}\sum_{i=1}^{2^{[n/2]}}L_D(\psi_i,\bar{\psi}_i,\overline{D}) \quad (37)$$

Thus one obtains the interaction between spinor fields ψ_i, $i=1,2,\ldots,2^{[n/2]}$ and gravitation and Yang–Mills fields in the usual way. It is worth noticing that all fermions ψ_i have the same mass m. If one assumes that all elementary particles get their masses due to Higgs' mechanism one may put $m=0$. Thus we deal with massless fermions. In some cases it is possible to incorporate Higgs fields into Yang–Mills fields with some symmetries over a space-time with extra dimensions (Forgass et al., 1980; Manton, 1979; Mayer, 1981; Mecklenburg, 1981; Witten, 1977). Thus it seems possible to obtain fermions ψ_1 with different masses. This will be done elsewhere.

Now we deal with Yang–Mills fields and should work with a concrete useful representation of Γ^A. We will consider the cases $n=2l$ and $n=2l+1$ separately.

If we suppose that the group G is a gauge group which unifies electromagnetic, weak, and strong interactions, then G has a subgroup $U(1)$ corresponding to electromagnetic interactions after breaking the symmetry. Let $\dim G=2l+1$ and let the parameter of electromagnetic subgroup $U(1)$

correspond to $A = n+4 = 2l+5$. Then we turn to the additional term in the Lagrangian (35) and perform the dimensional reduction for $b = n+4 = 2l+5$. One easily gets

$$i\varepsilon\frac{\sqrt{G}}{4c}\hbar H^{\alpha\beta}_{2l+5}\overline{\Psi}\Gamma^{2l+5}\left[\Gamma_\alpha,\Gamma_\beta\right]\Psi\eta = 2\varepsilon i\frac{l_{\mathrm{Pl}}}{\sqrt{\alpha}}q\sum_{i=1}^{2^{[n/2]}}F^{\alpha\beta}_{2l+5}\bar{\psi}_i\gamma^5\sigma_{\mu\nu}\psi_i\eta$$

where $F^{\alpha\beta}_{2l+5} = F^{\alpha\beta}$ (electromagnetic field). Thus we get for all fermions an electric dipole moment of the order (36) (Kalinowski 1981a, 1981b). If $\dim G = 2l$ then this term is forbidden and we have no fermion electric dipole moment. If such electric dipole moment exists then it means the unified gauge group G as an odd number of dimensions. In Borowiec (1979), Kerner (1980), and Mecklenburg (1982), one considered Dirac fields on a many dimensional manifold of the Klein–Kaluza type. But unfortunately in these approaches fermions possess minimal masses about 1 μg (Planck's mass terms) as in Thirring (1972). Our approach avoids these troubles as in Kalinowski (1981a,b). In Mecklenburg (1982) it is possible to cancel Planck's mass term. His approach really differs from ours and fails in the five-dimensional case.

5. DISCRETE TRANSFORMATIONS ON P

Now let us consider operations of reflections defined on the manifold P. To perform this we choose a local coordinate system on P:

$$X^A = (X^\alpha, X^a), \qquad X^\alpha = (\mathbf{X}, t) \tag{38}$$

Then $\Psi(p) = \Psi(X^A) = \Psi((\mathbf{X},t), X^a)$ and define transformations: space reflection Π, time reversal T, charge reflections C, and combined transformations ΠC, $\theta = \Pi CT$ in the following way:

$$\Psi^C(X^\alpha, X^a) = \overline{C}\Psi^*(X^\alpha, -X^a) \tag{39}$$

where $\overline{C}^{-1}\Gamma_\mu\overline{C} = -\Gamma^*_\mu$.

It is easy to see that

$$\overline{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = C\otimes\prod_{i=1}^{[n/2]}\sigma_1 \tag{40}$$

where σ_1 is a Pauli matrix and C is an ordinary charge conjugation matrix on E (see Appendix). Performing the dimensional reduction (21) one gets

$$\psi_i^C(X^\alpha) = C\psi_i^*(X^\alpha), \qquad i = 1, 2, \ldots, 2^{[n/2]} \tag{41}$$

and all color charges connected with Yang–Mills fields change signs. In Thirring (1972), Kalinowski (1981a, b), Rayski (1968) a similar problem was considered in the five-dimensional (electromagnetic) case. The reflection in coordinate X^5 was interpreted as an electric charge conjugation. For the space coordinate reflection we have

$$\Psi^\Pi(X^\alpha, X^a) = \Gamma^4\Psi(-\mathbf{X}, t, X^a) \tag{42}$$

Performing the dimensional reduction (21) one gets (see Appendix)

$$\psi_i^\Pi(\mathbf{X}, t) = \gamma^4\psi_i(-\mathbf{X}, t) \qquad i = 1, 2, \ldots, 2^{[n/2]} \tag{43}$$

i.e., a normal parity operator on E. For the transformation of time reversal T we have

$$\Psi^T(\mathbf{X}, t, X^a) = \bar{C}^{-1}\Gamma^1\Gamma^2\Gamma^3\Psi^*(\mathbf{X}, -t, -X^a) \tag{44}$$

Performing the dimensional reduction (21) one gets (see Appendix)

$$\psi_i^T(\mathbf{X}, t) = C^{-1}\gamma^1\gamma^2\gamma^3\psi_i^*(\mathbf{X}, -t) \qquad i = 1, 2, \ldots, 2^{[n/2]} \tag{45}$$

and all color charges connected with Yang–Mills fields of gauge group G ($\dim G = n$) change sign, i.e., a normal time-reversal operator on space-time. For the transformation $\theta = \Pi CT$ we put

$$\Psi^\theta(\mathbf{X}, t, X^a) = -i\Gamma^{2l+5}\Psi(-\mathbf{X}, t, X^a) \tag{46}$$

where $l = [n/2]$ and $\Gamma^{2l+5} = \gamma^5 \otimes \prod_{i=1}^{[n/2]}\sigma_1$ (see Appendix A). Performing the dimensional reduction (21) one gets

$$\psi_i^\theta(\mathbf{X}, t) = -i\gamma^5\psi_i(-\mathbf{X}, t) \tag{47}$$

For the transformation ΠC one gets

$$\Psi^{\Pi C}(\mathbf{X}, t, X^a) = \Gamma^4\bar{C}\Psi^*(-\mathbf{X}, t, -X^a) \tag{48}$$

Performing the dimensional reduction one gets

$$\psi_i^{\Pi C}(\mathbf{X}, t) = \gamma^4 C\psi_i^*(-\mathbf{X}, t), \qquad i = 1, 2, \ldots, 2^{[n/2]} \tag{49}$$

and all charges change sign.

It is clear that transformations obtained by us here do not differ from those known from the literature.

The additional term in Lagrangian (25) (in the case $n = 2l + 1$) breaks symmetry ΠC or T in an analogous way as in the five-dimensional case Thirring (1972), Kalinowski (1981a, b). This can be easily seen by acting on the Lagrangian with operator ΠC defined by (48).

APPENDIX A

In this Appendix we deal with the Clifford algebra (Atiyah et al., 1964; Cartan, 1966) $C(1, n+3)$. Owing to decomposition rules for $C(1, n+3)$ we write down a useful representation for Γ^A in terms of γ_μ. It is well known that any Clifford algebra can be decomposed into a tensor product of the four elementary Clifford algebras (Atiyah et al., 1964; Cartan, 1968):

$$\begin{aligned} &C(0,1) = \mathbf{C} - \text{complex numbers} \\ &C(1,0) = R \oplus R \\ &C(0,2) = H = \text{quaternions} \end{aligned} \tag{A1}$$

We have

$$C(1, n+3) = C(0,2) \otimes C(1, n+1) \tag{A2}$$

Because we deal with dimensional reduction to space-time E we define Clifford algebra $C(1,3)$ and we easily get

$$\begin{aligned} C(1, n+3) &= \left(\prod_{i=1}^{[n/2]} \otimes C(0,2) \right) \otimes C(1,3) \\ &= \left(\prod_{i=1}^{[n/2]} \otimes H \right) \otimes C(1,3) \end{aligned} \tag{A3}$$

It is well known that either

$$C(1, n+3) = C(1, n+4) \qquad (\text{iff } n+3 = 2l,\ l \in N_1^\infty) \tag{A4}$$

or

$$C(1, n+2) \simeq C(1, n+3) \qquad (\text{iff } n+3 = 2l+1,\ l \in N_1^\infty)$$

Let $\gamma_\mu \in \mathsf{L}(\mathbb{C}^4)$ $\mu = 1,2,3,4$ be Dirac matrices obeying conventional relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{A5}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, -1, -1, +1)$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \gamma_5^2 = -1$$

and let $\sigma_i \in \mathsf{L}(\mathbb{C}^2)$, $i = 1,2,3$ be Pauli's matrices obeying conventional relations as well:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{A7}$$

$$[\sigma_i, \sigma_j] = \varepsilon_{ijk}\sigma_k \tag{A8}$$

We introduce also the following notations: $\mathbb{1} \in \mathsf{L}(\mathbb{C}^2)$ is a 2×2 unit matrix and $\mathfrak{I} \in L(\mathbb{C}^4)$ is a 4×4 unit matrix. Thus one performs the decomposition (A3) and easily gets

$$\Gamma^\mu = \gamma^\mu \otimes \left(\prod_{i=1}^{[n/2]} \otimes \sigma_1 \right) \tag{A9}$$

or

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \tag{A10}$$

For $A \neq \mu$ one gets (in the case $n = 2l$):

$$\Gamma^{2p+1} = i\mathfrak{I} \otimes \left(\prod_{i=1}^{p-2} \otimes \mathbb{1} \right) \otimes \sigma_3 \otimes \left(\prod_{i=1}^{l-p+1} \otimes \sigma_1 \right)$$

$$\Gamma^{2p+2} = i\mathfrak{I} \otimes \left(\prod_{i=1}^{p-2} \otimes \mathbb{1} \right) \otimes \sigma_2 \otimes \left(\prod_{i=1}^{l-p+1} \otimes \sigma_1 \right) \tag{A11}$$

where $4 < 2p+1 < 2p+2 \leqslant n+4 = 2l+2$.

In the case $n = 2l$ we define also a matrix:

$$\Gamma^{n+5} = ii^3(l+1) \prod_{A=1}^{n+4} \Gamma^A = (\gamma^5) \otimes \left(\prod_{i=1}^{l} \otimes \sigma_1 \right) = \Gamma^{2l+5} \tag{A12}$$

or

$$\Gamma^{n+5} = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \tag{A13}$$

where $n = 2l$, $l \in N_1^{\infty}$.

If $n = 2l + 1$ we have $\tilde{\Gamma}^A = \Gamma^A$, $A = 1, 2, \ldots, 2l + 4$:

$$\tilde{\Gamma}^{n+4} = \Gamma^{2l+5} = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \tag{A14}$$

It is easy to check that

$$(\Gamma^{2l+5})^2 = -1 \text{ and}$$

$$\{\tilde{\Gamma}^A, \Gamma^{2l+5}\} = 0 \qquad \text{for } A \neq 2l+5 \tag{A15}$$

$$B = \bar{B} \otimes \left(\prod^{[n/2]} \otimes \sigma_1 \right), \qquad \gamma^{\mu +} = \bar{B} \gamma^{\mu} B^{-1} \tag{A16}$$

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